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ON THE TREATISE OF IBN YALB: MIRĀT AL-ḤISĀB*

HANIFA R. MUZAFAROVA

In the manuscript collection of the Institute of Oriental Studies (Academy of Science, Uzbek. S.S.R.) there is a bound volume with No. 6230/1, comprising several manuscripts written in Persian and Arabic. This volume is included in the list of Mathematical and Astronomical Manuscripts prepared by G.P. Matvievskaya.[1] The volume consists of 178 folios, the various manuscripts of which can be read with great difficulty because they are written diagonally in extra-ordinary small letters. The pagination has been done on folios with pencil. This collection has got a few works on mathematics and astronomy by different authors. The first one is *Mirāt al-Ḥisāb* by Ibn Yalb. It is written in the Persian language and covers 43 folios. The second work, which is astronomical, is written in Arabic, with 11 folios (43b—54a). The author of the work is Muḥammad Ibn Aḥmad. Nothing is known to us about his life.

In the third anonymous work (folios 54b-63a), rules of arithmetical operations on integers: doubling, halving (bifurcation), addition, subtraction, multiplication, division are explained in detail. A few methods of multiplication, having special names, have also been examined, e.g., *al-Ḍarb al-Qā'im, al-Ḍarb bi-Ṭul al-Tawḍīḥ, al-Ḍarb bil Aṣfār, al-Ḍarb al-Jawwāl, al-Ḍarb al-Mu^carraq,* [*Ḍarb*] *al-Muḥadhat, al-Ḍarb al-Qismah.*[2]

The fourth work of the above-mentioned collection is also written in the Arabic language and is a mathematical encyclopaedia, *Khulāṣat al-Ḥisāb* of the well-known scholar of the XVI century, Bahā' al-Dīn al-ʿĀmilī (1546-1622). The scribe was al-Sharīf Ṣāʿid Ibn Khwāja

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* Translated from the Russian original by Shameem Bano. The words in [ ] are our own additions for the sake of clarity, not given in Russian orginal.

1 Matvievskaya, p. 54, see also Matvievskaya and Talashev, p. 54.

2 See Ahmadov, p. 89, for these methods and concerning operations with even numbers.

Muḥammad. This treatise had become very famous in the countries of the East during XVIII-XIX centuries.[3a]

In the present article we describe the first work of the aforementioned volume: *Mirāt al-Ḥisāb* by Ibn Yalb; nothing is known about his life and works.

Ibn Yalb describes important rules of arithmetic and algebra. In the section on geometry, he treats simple geometrical figures, such as the circle and its parts, moon-shaped figure, colza, lens, trapezium, cogged wheel, drum-shaped, almond, triangle etc. The next part of the manuscript is devoted to the theory of inheritance: "Inheritance rights of *Shaykhzāde*", and "the science of inheritance ʿUlūm". It is said at the end of both sections that a commentary is written on the work of Ibn Ḥājī Muḥammad al-Baysun of Sharīfzāde family which used to reside near the Salt river of India in 1231 A.H. This remark is [presumably] based on the authority of al-Bukhārī.[3b]

According to the date given in the beginning of the manuscript, 1098 A.H. (1687/88 A.D.), the author lived in XVII century. It is known that during XVII-XIX centuries, the level of the development of science in the East had a sharp fall as compared to the XV century, because of unpleasant changes in the political and economic life of the people. Since concrete details of mathematics of that period are poorly known, it is necessary to study manuscript sources, so that its level could be evaluated objectively, on the basis of source-material research.

In the preface of *Mirāt al-Ḥisāb,* it is said that the work contains a translation of the mathematical part of the astronomical work *Siddhānta Śiromaṇī* by the Indian scholar Bhāskara II (1115-1183). This Siddhānta (Crown of Science) was written in 1150, in the Sanskrit language.[4] Bhāskara was also the author of six other works. Two Sanskrit treatises are specially devoted to mathematics: *Līlāvatī* and

3a The Arabic text with its translation has been published in European languages. In 1843, G. Nesselman published a text with German translation and in 1846 a French translation by A. Marr, see Sirajdinov, p. 190. A large number of manuscripts of al-ʿĀmilī's treatise are available in different libraries of the world; see Ibadov and Ahmedov. For instance, Manuscript Division of State Firdausi Library, No. 1260, 1788, 1239, 1259. Manuscript No. 1260 was copied by Muḥammad Alamīr in 1241 according to colophon. Manuscript No. 831 of the Academy of Science (Tajik S.S.R.) has been investigated by G. Sobirov, (pp. 5-16). Several commentaries on the work of Bahāʾ al-Dīn are also well-known.

3b This sentence is not very clear. Therefore a free translation is given here. But it is important, since the year 1231 A.H./1816 A.D. indicates that the scribe copied this treatise after 1816 (SMRA).

4 Volodarsky, p. 7.

Bījagaṇita. The manuscript of *Mirāt al-Ḥisāb* by Ibn Yalb is based on the Persian translation of those works.[5a]

Līlāvatī (meaning beauty) is mainly concerned with arithmetic. The title of the book alludes either to Bhāskara's daughter or to the (science of) mathematics itself. It is divided into 13 chapters. In the first chapter metrological tables are given. In the following chapter operations with whole numbers and fractions including extraction of square and cube roots are described. In the third chapter the solution of arithmetical problems with the help of method of inversion, rule of one wrong position and other methods have been given. The fourth chapter is devoted to the problems of reservoir and the fifth to the summation of a few arithmetical series. The sixth chapter is confined to the planimetry. Seventh to eleventh chapters are devoted to geometry, mainly to figures, to the problems of the measurement of volumes, to the so-called Diophant analysis, and to detailed investigations on algebra. In the last chapter, a series of problems on combinatorics have been examined.

Bījagaṇita consists of eight chapters and it is a study of algebraic solution of the first and second degree equations.[5b] In the first chapter rules of operation with positive and negative numbers have been formulated; the second and third chapters deal with rules for the integral solution of indefinite equations of the first and second degree. After that follows the solution of problems of linear equations with a single and several unknown variables (IV chapter) and quadratic equation (V chapter). A few geometrical questions and two proofs of Pythagoras' theorem have also been described here. In the sixth chapter different kinds of definite and indefinite linear equations with few unknown variables have been collected. The seventh and eighth chapters consist of additional material on indefinite equations of second degree. In 1813 A.D. Edward Strachey published the English translation of algebraical treatise of *Bījagaṇita.* This translation has been done not from the original but from the Persian translation of 1634.[6]

In the preface of Ibn Yalb's treatise, it is said: "According to necessity, the obedient, uneducated, ignorant slave Ibn Yalb, after learning the works of mathematicians, astronomers and distinguished geometricians, presents this information that in ancient times, one of the best astrologers and excellent representatives of Indian science in

5a For an account of Bhāskara's work, see Yushkevitch and Volodarsky [see also Editorial Comments].

5b Brahamāgupta, the distinguished scholar of VII century also dealt with this topic using quite a developed symbolic (SMRA).

6 Volodarsky, p. 4. The Persian translation was done by ʿAṭāullāh Rāshidī (17th c) —SMRA.

the field of celestial astrology (*tafhīm falakī*), Bhāskara had written a book in Sanskrit *Sahkarat,* which contains studies on celestial spheres, on conjunction of planets and fixed stars, on the nature of celestial sphere, and named it *Siddhānta Śiromaṇī* (the Crown of Science) and divided it into three chapters."

"The first chapter is *Karha Katat* (the science of skies), the second chapter *Līlāvatī* (meaning beauty) is devoted to arithmetic. It describes the rules of multiplication, division and calculation of area. The third chapter, *Bījagaṇita*, gives an account of the science of *al-jabr* and *al-muqābalah.*"

"As the readers of Persian language could not use this work, this insignificant slave, after learning the subject from the astrologer Kāmil ᶜĀmil Makhdūm, translated it into Persian language, supplemented several sections such as the ones on multiplication, division, duplication, extraction of a square root, duplication of a cube, simplification of numbers and determination of areas etc. The aforementioned topics were not described in detail in the *Siddhānta.* So, in this way, the obedient slave prepared this treatise and gave it the title : *Mirāt al-Ḥisāb.* There is no necessity of writing another such work, the obedient slave in advance requests and thanks those readers, who may correct his work, discuss it in a wise circle and may do the correction with their own pen".

The following eight operations are attributed to the Indian mathematicians of 5th-12th centuries, as the basic arithmetical operations: addition, subtraction, multiplication, division, duplication of a square, extraction of a square root, duplication of a cube and extraction of cube root. As it is noted by Volodarsky,[7] the majority of these operations are not considered in *Siddhāntas.* Āryabhaṭa (V-VI century) gives only rules for the extraction of square and cube roots and Brahmagupta starts only with a rule for the extraction of cube root. That is why, after examining the given manuscript in detail, it is concluded that these operations could be supplemented by its author.

Let us discuss the content of the treatise of Ibn Yalb in brief: In the beginning, he describes *Līlāvatī.* The first chapter is called "calculation and debt", (*ḥisāb wa qarḍ*). It consists of five small sections. Rules of operation with positive and negative numbers have been described here in detail. As it is well-known, in [the history of] mathematics, negative numbers were first traced in the 8th book of ancient Chinese treatise: *Mathematics in Nine Books,* in which rules of addition and subtraction of negative numbers have been formulated.[8]

7 Volodarsky, p. 31.
8 Ibid, p. 87.

Indian mathematicians, starting with Brahmagupta systematically used the negative numbers, interpreted the positive numbers as "property" or "object" and negative numbers as "debt". In the manuscript of Ibn Yalb, a special symbol has been introduced for a negative number. It is a dot, which has to be put on the top of the numeral. Other Indians have also done the same. Example: 3 is object, •3 is debt. Adding the "object" and its equivalent "debt" gives zero. Example:[9] $3+{*3}=0$.

While explaining the rules of addition of numbers, the author writes: "you must know that if object is combined with object and debt with debt, the sum of objects will be an object and sum of debts will be a debt. And if object is combined with debt or debt with object, then the surplus shall be object."[10]

Further, the rule of addition of two numbers of many digits by usual method is given and illustrated with examples. For instance the addition of positive numbers,

26075	i.e.	26075
9834		+9834
35909		35909

No sign of addition is employed. Moreover, rules of subtraction of numbers are given for four cases:

Case I: "From a large number we subtract a small one, the result will be the object. While subtracting a certain number from another, we write the minuend above the subtrahend, draw a horizontal line and take away digit for digit, the subtrahend from the minuend. And if something is left, we shall write the ramainder underneath, and if nothing is left, we write the zero. With this, the operation comes to an end."

Example: If we want to subtract 963 from 8045, we shall write them as given below. The sign of the negative number (sign of debt) should be put on the subtrahend:[11]

Object	8045		8045
		i.e.	
Object	*963		—963
	7082	remainder	7082

9 Ibn Yalb, f. 1 b. Hereafter the terms object and debt will not be in quotes. Further due to technical reasons we shall use an * instead of a dot (Ed.).

10 Ibid f. lb. In fact the result could be object or debt as the case may be (Ed.).

11 Ibn Yalb, f. 2a.

Case II: If the subtrahend and minuend are debt, then we drop the sign of the negative number from the minuend. Thereafter the operation is carried out, and we put the sign of the debt in the result of subtraction

Case III: If the minuend is debt and the subtrahend is object, then we drop the sign of negative number from the minuend, further we add both numbers:[12]

Example:

*8045		8045
963	i.e.	—963
9008		9008

Case IV: And if the minuend is object, and the subtrahend is debt, then the sign of the negative numbers is to be applied to the minuend, thereafter we add the result and denote it as debt as follows:[13]

8045		—8045
*963	i.e.	—963
9008		—9008

[In case III and IV, the signs of the result are incorrect; they should be negative in III and positive in IV—SMRA]

Indians also used to denote the subtraction by a point on the subtrahend or by the sign + marked thereafter.[14]

The following section is devoted to multiplication. The rule of multiplication and division of negative numbers, is found in the work of Bhāskara II for the first time. Probably it was devised by his predecessors. In any case, Bhāskara goes further than the question of the extraction of square root from positive numbers.[15]

Ibn Yalb presents this part without any change. The rule of multiplication is given in a general form for positive as well as for negative numbers. He writes: "You must know, if multiplicand and multiplier both are objects or both debt then the result of multiplication in both cases will be object. And if the multiplicand is debt then the result of multiplication will be debt."[16]

12 Ibid, f. 2a.
13 Ibid, f. 2a.
14 Volodarsky, p. 50.
15 Yushkevitch, p. 140.
16 Ibn Yalb, f. 2a.

Further, he gives a method of multiplication for a number of many digits, which is identical with the modern procedure of multiplication:

Example:

$$\begin{array}{r} 9045 \\ \times 963 \\ \hline 8710335 \end{array}$$

Going to division, Ibn Yalb defines this operation according to Euclid and then examines the case of the division of positive and negtive numbers. He writes: "The division of one number by another gives a third number, such that if it is multiplied by the second, the first number is obtained. The first number is known as dividend, the second the divisor and the third the quotient."[17]

"You must know, that if the divisor and dividend are objects or debts, then in both cases the quotient will be object.[18a] If one of these two will be debt, then the quotient will also be debt, i.e., if the divisor is object, and the dividend is debt and vice versa, then in both cases the quotient will be debt".[18b]

Further the rule for the division of many digited number by another number of many digits is treated in detail and corresponding examples are given. For instance, $235074 \div 579 = 406$.

The next section carries the title: "On raising the power and extracting the root". The treatment of these arithmetical operations by mathematicians of India, Near and Middle East is described in the works of A.P. Yushkevitch.

The Sanskrit term for the root is *mula* and *pada*. The actual meaning of *mula* is the root of a tree. The Arabic word *jadhar* is also actually a translation of "the root of a tree". The modern term for root is the Latin word *radix*. It is the translation of the Arabic *jadhar*, signifying the root and base of a square and in turn is translated from the Sanskrit *mula*.

The earliest description of the procedure to extract square root is found in the works of Āryabhaṭa. In fact, it were the Chinese who introduced the method for the extraction of root to the Indians who brought about considerable change in it.[19a] Methods for the extraction of root for the second and third powers are found in arithmetical

17 In the Russian original the words: dividend (*delimoe*) and divisor (*delitel'*) are inadvertantly interchanged in this sentence (Ed.).

18a In the Russian original it is wrongly stated that "it will be debt" (Ed.).

18b Ibn Yalb, f. 2b.

19a This assertion of Muzafarova is questionable (Ed.).

works of al-Kās͟hī (15th c.), al-Nasāwī (11th c.), Naṣīr al-Dīn Ṭūsī (13th c.) etc.

In the manuscript, Ibn Yalb applied the operations of squaring and extraction of square root to positive as well as to negative numbers. The author writes: "If every number is multiplied by itself then the number obtained is called the square (*majd͟hūr*) and the number itself is called the base (the root, *jad͟har*). One must know that the square of an object is an object and the square of a debt is also an object. For example: If the number 3 is object or debt, by squaring an object is obtained. Also one must know that the square root of an object will be an object and/or a debt."[19b]

*3 *3 9	3 3 9

Furrher, the method of extracting the square-root of a positive number (object) has been described. Examples are given and the method of solution is described in detail.

In the middle of the ninth century, the two values of a square root were fully well-known to Indian mathematicians. Mahāvīr writes: "Square of a positive or negative number is positive and their square-roots will correspondingly be positive or negative." Mahāvīr also writes on the impossibility of extracting the square-root of a negative number. "As the negative number by its very nature does not correspond to a square, so it does not possess a square-root."[20]

The second chapter consists of two parts:

(i) To get a cube, the rule is based on the formula:

$$p^3 = p.p.p = (a+b)^3 = a^3+3a^2b+3ab^2+b^3$$

(ii) The extraction of the cube-root of a number: As it is well-known, the method of extraction of a cube root is identical with the method of Chinese, i.e., the method which is now known as the method of Ruffini-Horner, first described by al-Nasāwī (12th c.).[21]

Abul Wafā (940-998) has written on the extraction of roots of 3rd, 4th and 7th degrees. These works have not been discovered. K͟hayyām's treatise: "Difficulties of Arithmetic" (*Mus͟hkilāt al-Ḥisāb*)

19b Ibn Yalb, f. 2b.
20 Volodarsky, pp. 86-87.
21 Matvievskaya, p. 134.

has also been lost, in which he had given the method of determination of integral root of any natural power of an integer. The extraction of the cube-root of integers is formulated by Ibn Yalb after Indian mathematicians. This is illustrated in the example:

$$\sqrt[3]{51478848}$$

The third chapter is on the application of zero. Indian mathematicians considered the zero as a number. They worked out its arithmetical opertions.[22] They described the properties of zero and cite oral rules:

$$a+0=a,\ 0+a=a,\ a—a=0,\ a:0=0,\ 0:a=0.$$

In the beginning, the division of a number different from [not equal to —Ed.] zero by zero was thought by Indians to be impossible, but later, they concluded that the division by zero is infinity. Bhāskara II has written that the value of [the quantity], such as, a/0 (where a=0) does not change whatever is added to zero or subtracted from it. We do not find the operation with zero in well-known works of mathematicians of Central Asia.[23]

Following the Indian rules, Ibn Yalb describes operation with zero in detail. He subdivides these rules according to form (*naw*ᶜ), firstly related to the rules of addition and subtraction, and secondly to the rule of multiplication, division, squaring and extraction of square-root. Ibn Yalb divides this chapter into two parts: The first part consists of two sections:

(i) *Addition:* "One must know that by adding zero to a number, one obtains the same number. For example: 3+0=3, 0+3=3."

(ii) *Subtraction:* "One must know that if we subtract a number from zero, then in the result only the sign of the number changes to the opposit one, i.e., the object turns into debt and debt into object. For example, if number 3 is to be subtracted from zero, it cannot be subtracted, but turns into debt. If number 3 is debt and we subtract it from zero, it cannot be subtracted, but turns into object".[24]

The second part consists of four sections.

(1) *Multiplication:* "One must know that if a number is multiplied by zero, then this number vanishes and the result obtained is zero. For example: If 2 is multiplied by zero, then it is equal to zero".

22 Cf. Yushkevitch. Unfortunately it is not clear from the Russian original whether the rule given above is by Ibn Yalb (i.e. it is a translation from *Līlāvatī*) or not (Ed.).

23 Cf. Volodarsky.

24 Ibn Yalb, f. 4a.

(2) *Division:* "One must know, that if zero is divided by a number, the quotient will be zero, and if a number is divided by zero, then the denominator will be violent (*makhraj al-ghadd*), $(a/0=0)$. If zero is divided by 3, then zero is obtained. If 3 is divided by zero, then one obtains zero in the denominator, which is called in Sanskrit *Antatrāshī*. It means if its square roots is extracted, then there appears invalidity (بغيض)."

(3) *Squaring:* The author writes: "One must know that if zero is raised to the power 2, then zero is obtained, $0^2=0$".

(4) *Extraction of a Square Root:* "One must know that if the square-root is extracted from zero, then zero is obtained, $\sqrt{0}=0$.[25]

The value of a number divided by zero (i.e. $a/0$) for $a=0$ would not change whether [some thing] is added to or subtracted from it.

The fourth chapter is devoted to the arithmetic of fractions. It consists of six parts.

In the first part general information about fractions is given. Fractions are written as in now-a-days. Numerator is above the denominator, only without a dividing line. Indians have also denoted fractions in the same way.[26] Fractions are separated from each other by vertical lines. For example:

$$\left|\begin{matrix}1\\2\end{matrix}\right| \qquad \left|\begin{matrix}1\\6\end{matrix}\right| \qquad \left|\begin{matrix}2\\3\end{matrix}\right|$$

In a mixed fraction the integral part is written above the fraction, for example as:

$$\begin{matrix}6\\3\\4\end{matrix}$$

Further the rule of transforming mixed fractions into improper fraction is given. "For the transformation of mixed fraction into improper fractions, the integral part is multiplied by the denominator and the result of multiplication is added to the numerator of the fraction".

In the second part the addition of fractions has been described. It is said that for adding fractions with same denominators, we add the numerators of fractions and write the result as numerator but we carry over the denominator of the fraction without any change. Then, if the numerator is more than the denominator then we divide the numerator by the denominator and so we pick out the integral part of the fraction.

25 Ibn Yalb, f. 4a.

26 Sirajdinov and also Matvievskaya.

If the numerator is equal to the denominator, then the fraction is turned into an integer. If the numerator is less than the denominator then the fraction will be left without any change. We want to add fractions (3/4) and (5/4), it is written as,[27]

3 4	5 4	8 4

From this entry it is clear that the sign of addition is not cited, besides [just] the writing down of fractions means addition. For example:

$$(a/b)+(c/b)=(a+c)/b$$

has to be written as:

a b	c b	a+c b

Examples on addition of fractions are:

1 2	1 6	2 3

18 36	6 36	24 36

6 12	2 12	24 36

1 2	2 12	4 6

[The last two "frames" have not been explained by the author —Ed.]

The third part is devoted to subtraction of fractions, which is denoted by using a point. For example:

2 3	*1 6	3 6

The rule of multiplication described in the fourth part is as follows. "For the multiplication of fraction, we multiply the numerators and write [the product] as numerator, also the product of the denominators, we write as denominator.

The rule of division of fractions (fifth part) is also identical with the rule of the division of fractions in modern arithmetic.

In the sixth part of the fourth chapter, rules for squaring of fractions and for the extraction of square-root of fractions have been described. The author writes: "The square of a fraction is the square

27 Ibn Yalb, f. 4b.

of the numerator divided by the square of the denominator. For example:

4	16
3	9

3	9
2	4

The square-root of a fraction is equal to the square-root of the numerator, divided by the square-root of the denominator.[28]

The fifth chapter of the manuscript of Ibn Ẏalb is devoted to algebra. Its title is "On the rules of determination of unknown quantities". This chapter consists of nine sections.

In the manuscript, it is stated as follows: "To denote the unknown in mathematics, Indian letters are used. These letters cannot be used by Persian-knowing nations, so we have adopted [our] notation: One unknown is denoted by the *abjad* letter *shīn* (ش) the other by letter *mīm* (م), a known number by the letter *ᶜayn* (ع)."[29]

In the words of A.P. Yushkevitch, in almost all the works of mathematicians of Islamic countries whose works have been transmitted to us, algebraic symbols are completely absent. This remark applies, in every sense, to all Oriental scholars starting from al-Khwārizmī to al-Kāshī. However in the Arabic West we come across a symbolism in the arithmetical-algebraical treatise of Abul Ḥasan ᶜAlī Ibn Muḥammad al-Qalāsadī (15th c.) "Revealing the Science of Ghubār" (كشف المحجوب من علم الغبار).

Therein the square root is denoted by the first letter of the word *jadhr* (root) and which was written above the number and the same sign serves for denoting the unknown in the rule of three (proportions). Qalāsadī has a sign for equality. The symbolism of al-Qalāsadī is so much developed that it is not known [can not be presumed] whether he had invented the whole of it by himself.[30] In Europe, the development of algebraic symbolism started in the 15h century.

Ibn Yalb examined rules for operation with one and several terms having one unknown.

(1) Rules of addition and subtraction of algebraic expressions: addition and subtraction are carried out for one and the [same] type among unknown, i.e. (ش with ش) , square of this number with [its own] square. Subtraction is achieved in the same way.

(2) Rule of multiplication is formulated by Ibn Yalb as follows. "You must know that if a known number is multiplied by an unknown, then an unknown number is obtained".

28 Ibn Yalb, f. 5a.
29 Ibn Yalb, f. 5a.
30 Yushkevitch, p. 260.

(3) Rule of division of algebraic quantities: "For every unknown and known quantity, the divisor is multiplied correspondingly and is subtracted from the dividend so that nothing is left, and gradually the quotient is obtained", e.g.,

Dividend : *15 ش^2 *7 ع ش 2 ع^2
: $-15x^2 - 7ax + 2a^2$
Divisor : *2 ع 3 ش = $-2a+3x$

Here ع is denoted by a and ش by x.[31]

(4) Rules of squaring: Ibn Yalb explains it as follows: "For squaring of unknown and known quantities, it is necessary to multiply the base by itself".[32] For example in modern notation we can write [his example] as follows:

$$(6a-4x)^2 = 36a^2 - 48ax + 16x^2$$

that is for squaring an algebraical expression the author uses the rule based on the formula:

$$(a+b)^2 = a^2 + 2ab + b^2$$

(5) Rules for the extraction of a square root: This rule is just the reverse of the rule of squaring. For this case, the same example given above is used in reverse order. That is: firstly the algebraic expression

36ع^2 *48ع ش 16ش^2

is given,[32a] and after extracting the square root, the result is given, i.e., 6ع *4ش. In orther words, it is shown that

$$\sqrt{36a^2 - 48ax + 16x^2} = (6a - 4x)$$

The sixth chapter is on the rule for operation with algebraic expression (مجهول) having different signs (*mukhtalif al-ʿilāmāt*).

1. *Addition and Subtraction:* Here a rule is given for the addition and subtraction of algebraic expressions, having different signs. For example, on folio 6a, the addition of the following expressions:

3ش 5ص 7ق
*2ش *3ص *1ق

leads to the result

1ش 2ص 6ق

and on subtraction, one obtains

5ش 8ص 8ق.

31 It is not clear whether this notation of squaring a letter-symbol by writing 2 as exponent is given in the Ms. of Ibn Yalb or not. In fact it should be

*15ش مر *7ش ع 2ع مر

where مر stands for *māl* meaning square as was done by al-Qalāsadī (SMRA).

32 Ibn Yalb, f. 6a.

32a Ibid.

This is exactly the same in our modern notation, with ش fot x, ص for a and ق for y.[33]

2. *Multiplication, division, squaring* and *extraction of square-root:* For each of these operations, corresponding rules and examples have been given.

In the seventh chapter Ibn Yalb deals with rules of operations of irrational quantities (*karaṇa*). The author writes: "You must know that *karana* is an irrational quantity (اصم).[34] The Arabic-Persian term for the irrational quantity is *aṣm* which literally means deaf. *Karaṇa* is a special term for root, introduced by Bhāskara II: ka 9, ka 450, ka 75, ka 54 denotes

$$\sqrt{9} + \sqrt{450} + \sqrt{75} + \sqrt{54}.$$

Writing of the irrational quantities in this manuscript coincides completely with that of Bhāskara. For example: The expression 52 اصم 175اصم 45اصم 13اصم 3اصم 2اصم, as given in the manuscript, denotes:

$$\sqrt{52} + \sqrt{175} + \sqrt{45} + \sqrt{13} + \sqrt{3} + \sqrt{2}.$$

Further, a definition of an irrational number is given which is as follows: "The square root of a number is called irrational, if it cannot be extracted exactly."[35] He describes square and cube roots of numbers.

In this chapter, methods of addition, subtraction, multiplication, division, squaring and extraction of square-root of irrational quantities have been given in detail. For all those operations, rules and corresponding examples have also been given. In modern notations they are as follows:

1. $\sqrt{(a+\sqrt{b})} = (1/\sqrt{2})[\sqrt{(a+\sqrt{a^2-b})} + \sqrt{(a-\sqrt{a^2-b})}]$
2. $\sqrt{a+b+2\sqrt{ab}} = (\sqrt{a}+\sqrt{b})$

Examples are given for those cases in which these formulas are applied and the expressions are simplified.[36]

As it is known, starting from the work of al-Khwārizmī, all algebraic treatises have a section on rules of operation with surds: With multiplication and division in the beginning and then afterwards addition and subtraction. In examples, a numerical surd (irrationality) has the same role as a number—not only the surds themselves but also as coefficients of equations. The X book of Euclid's *Elements* is dedicated to numerical irrationalities.

33 In the Russian version addition and subtraction in modern notation are actually carried out, we are omitting them—Ed.

34 Ibn Yalb, f. 6b.

35 Ibn Yalb, f. 7a.

36 Ibn Yalb, ff. 7a-8a.

Ibn Yalb also describes the negative irrational quantity and denotes it by a point, which could be put on the root. "If an irrational quantity is a debt, then taking it as an object, we extract the square root and we make the result as debt." For example.[37]

$$-\sqrt{2} = -1.414$$

The eighth chapter is on the rules for operation of *kuṭak* (کوتک).[38] In the Indian language *kutak* means to crush, to grind, or to pulverise.[39] In this chapter information on the handling and solution of indefinite equations is given in detail.

The earliest information on the application of indefinite equations in Indian mathematics appeared in the middle of the first millineum B.C. in *Śulba Sūtra.* While transforming a square with the area a^2 into another square with the area ma^2, an equation of second degree ($x^2 - y^2 = z^2$) is to be solved, and its solution is given in the form:

$$m^2, \frac{m^2-1}{2}, \frac{m^2+1}{2}.$$

The indefinite equation became the object of a systematic study in the first century A.D., when Indian mathematicians and astronomers were confronted with a series of calendarico-astronomical problems.[40]

It is said in the manuscript: "For the rules of operations with *kutak,* there must exist "quantities": dividend (*maqsūm*), divisor (*maqsūm ʿilayhi*) and a constant term (*mazīd*). The dividend, divisor and the constant term are supposed to be divided by their common multiplier. And if dividend and divisor possess a common factor by which the free term cannot be divided, then this case by itself leads to an indefinite irrational quantity.[41]

Further, an example is cited for the case when GCM of all the three numbers exists, namely, for the numbers 221 and 195. The entry is as follows:

Dividend	221	free term	65	GCM 13
Divisor	195			

37 Ibn Yalb, f. 8b.

38 The Sanskrit word is actually *kuṭṭaka* (pulveriser). It is a well-known method of calculation (SMRA).

39 Volodarsky, p. 126.

40 Volodarsky, p. 126.

41 Ibn Yalb, f. 10b.

A method for the solution of indefinite linear equation is also given for the type

$$ax+b=cy$$

with a, b, c as positive integers.

In the ninth chapter with the title *majdhūri mazmar,*[42] the rules for the formation of second degree indefinite equations have been explained. The rules, which are given in the manuscript, concern the equation of the type.

$$ax^2 \pm b = y^2$$

where a is the multiplier, b the free term and y is the unknown number. The author considers both cases: for positive and negative [coefficients]. This rule is explained as follows: "Let us take an arbitrary number (object or debt) and obtain its square, then multiply it by the constant multiplier, and add to or subtract from the result the constant term. The number so obtained will be the square of the number. If the constant term is added to the result, then one equation is obtained, i.e. $ax^2+b=y^2$, and if it is subtraced, then one gets another equation, $ax^2—b=y^2$.[43] The given examples can be written down as follows:[44]

$$8x^2+1=3y^2$$
$$8x^2—1=17y^2+6$$

In the tenth chapter, the rule of three quantities (حساب سه رأس) is dealt with; it is about proportions. The terminology for the rule of three quantities (حساب سه رأس), is derived from Sanskrit. It means that the rule, which is followed for finding the number x, is generated by the three given numbers, a, b and c, of the proportion, i.e.,

$$a : b=c : x$$

The following examples are given by Ibn Yalb.[45]

$$12 : 40 = 3 : 10$$
$$40 : 30 = 50 : 16$$

The eleventh chapter deals with the determination of four types of areas: of a square, of a circle, of a right angled and of an arbitrary

42 Probably the title is *majdhūr-i muḍmar*. In the Russian original there are no diacritical marks (SMRA).

43 Ibn Yalb, f. 12b.

44 Ibid, f. 13a. In the Russian original, the l.h.s. of the second equation has a plus sign inadvertantly. (Ed).

45 Ibid, f. 16a. Note in the second line below, there is a mistake in the Russian original, it could be 40: 30=20: 15 (SMRA).

general triangle, and area of a quadrilateral. The formula of Heron is cited, namely:

$$S=\sqrt{(r-a)\ (r-b)\ (r-c)\ (r-d)}$$

where r is half of the perimeter and a, b, c, d are the sides.

The twelfth chapter is devoted to the summation of arithmetic and geometric progressions.

In the manuscript, numerical examples are given for the determination of the sums of series of natural numbers, as 1, 2, 3, 4,............ 10, and 1, 2, 3, 4,.........,9 according to formula:

$$S_n = (1/2)\ (1+n)n,$$
$$S^{10} = (1/2)\ (1+10)\cdot 10 = 55$$
$$S_9 = (1/2)\ (1+9)\cdot 9 = 45.$$

For the determination of the sum of a geometric progression also numerical examples are given. First part of *Mirāt al-Ḥisāb* called also *Līlāvatī* comes to an end with those examples.

Further the author introduces the second part of *Mirāt al-Ḥisāb* based on that work of Bhāskara titled as *Bījagaṇita.* He writes: You must know that this part is about the rules of *al-jabr wa al-muqābila* and it consists of four sections.

1. Linear equations with one unknown.
2. Linear equations with a few unknown.
3. Linear equations with those few unknowns which are obtained as the result of multiplication and division.

A few problems are given here for the solution of linear equations with one unknown. One of them is related to the determination of the price of a horse: "One (person) has 300 coins and 6 horses. Another (person) has 10 such horses, but he does not have even 100 coins. Both are equally rich. What is the price of a horse?"[46] It yields:

$$6x + 300 = 10x - 100,$$

x being the cost of a horse. It turns out that a horse costs 100 coins. This problem is given in the work of Bhāskara II.[47]

Problems reducing to the solution of a system of linear equations with two unknowns are cited on ff. 20a-22a. Further, another solved problem is given, which leads to the solution of a third degree equation with one unknown.[48] There are also problems, which lead to the solution of an irrational equation.[49]

46 Ibn Yalb, f. 19b.
47 Volodarsky, p. 89.
48 Ibn Yalb, ff. 22a-23b.
49 Ibid, f. 23b.

Moreover problems have been formulated for determining the perimeter, the area of a right angled triangle, the position of a median, the altitudes of triangles etc.

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* Unless otherwise stated all titles are evidently in Russian language (Ed.).